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Field fluctuations and macroscopic properties for nonlinear composites

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Abstract

A recently introduced nonlinear homogenization method [J. Mech. Phys. Solids 50 (2002) 737–757] is used to estimate the effective behavior and the associated strain and stress fluctuations in two-phase, power-law composites with aligned-fiber microstructures, subjected to anti-plane strain, or in-plane strain loading. Using the Hashin–Shtrikman estimates for the relevant "linear comparison composite," results are generated for two-phase systems, including fiber-reinforced and fiber-weakened composites. These results, which are known to be exact to second-order in the heterogeneity contrast, are found to satisfy all known bounds. Explicit analytical expressions are obtained for the special case of rigid-ideally plastic composites, including results for arbitrary contrast and fiber concentration. The effective properties, as well as the phase averages and fluctuations predicted for these strongly nonlinear composites appear to be consistent with deformation mechanisms involving shear bands. More specifically, for the case where the fibers are stronger than the matrix, the predictions appear to be consistent with the shear bands tending to avoid the fibers, while the opposite would be true for the case where the fibers are weaker.

Keywords: Nonlinear homogenization; Field fluctuations; Fibrous composites; Shear bands

1. Introduction

This paper is concerned with the application of the recently developed "second-order" homogenization method of Ponte Castañeda (2002a) to two-phase power-law composites with arbitrary heterogeneity contrast. One of the interesting aspects of this new method is that, unlike the previous version (Ponte Castañeda, 1996), it incorporates information about the fluctuations of the relevant fields, providing nonlinear estimates that are exact to second order in the heterogeneity contrast and that do not violate rigorous bounds.

For completeness and later reference, it is recalled here that bounds of the Hashin–Shtrikman type for nonlinear composites were first given by Talbot and Willis (1985), using a generalization of the variational

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principles of Hashin and Shtrikman (1962) for nonlinear media. More general types of bounds, including three-point bounds, were obtained by Ponte Castañeda (1991) by means of new variational principles (Ponte Castañeda, 1992) involving "linear comparison composites." Equivalent bounds for the special class of power-law composites were generated by Suquet (1993) using linear comparison composites and Hölder-type inequalities. For more comprehensive reviews of the nonlinear homogenization literature, the reader is referred to Ponte Castañeda and Suquet (1998) and Willis (2000). Field fluctuations in composites with linear elastic properties have been studied by Kreher and Pompe (1985) and Bobeth and Diener (1987), among others. Corresponding studies have apparently not yet been carried out for nonlinear composites in the mechanical context, although a start along this direction was given in Ponte Castañeda (2002a,b). There are also some recent results for weakly nonlinear composites (Pellegrini, 2000), as well as some theoretical results for strongly nonlinear composites (Pellegrini, 2001; Ponte Castañeda, 2001) in the context of conductivity.

2. Effective behavior

The assumption is made here that the material is composed of N different phases, which are *randomly* distributed in a specimen occupying a volume Ω, at a length scale that is much smaller than the size of the specimen and scale of variation of the loading conditions. The constitutive behavior of the nonlinear phases is characterized by *convex strain potentials* $w^{(r)}$ $(r = 1, \ldots, N)$, such that the local stress–strain relation is determined by:

$$\boldsymbol{\sigma} = \frac{\partial w^{(r)}}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}), \qquad w(\mathbf{x}, \boldsymbol{\varepsilon}) = \sum_{r=1}^{N} \chi^{(r)}(\mathbf{x}) w^{(r)}(\boldsymbol{\varepsilon}), \tag{1}$$

where the characteristic functions $\chi^{(r)}$ are 1 if the position vector $\mathbf{x}$ is in phase r and 0 otherwise. The effective behavior of the composite is characterized by the *effective strain potential*. Using the minimum potential energy principle it can be written as:

$$\widetilde{W}(\bar{\boldsymbol{\varepsilon}}) = \min_{\boldsymbol{\varepsilon} \in \mathscr{K}(\bar{\boldsymbol{\varepsilon}})} \langle w(\mathbf{x}, \boldsymbol{\varepsilon}) \rangle = \min_{\boldsymbol{\varepsilon} \in \mathscr{K}(\bar{\boldsymbol{\varepsilon}})} \sum_{r=1}^{N} c^{(r)} \langle w^{(r)}(\boldsymbol{\varepsilon}) \rangle^{(r)}, \tag{2}$$

where angular brackets $\langle \cdot \rangle$ and $\langle \cdot \rangle^r$ are used to denote volume averages over the composite (Ω) and over the phase r (Ω^r), respectively, $c^{(r)}$ are the volume fractions of the phases, and $\mathscr{K}(\bar{\boldsymbol{\varepsilon}})$ denotes the set of kinematically admissible strain fields, given by:

$$\mathscr{K}(\bar{\boldsymbol{\varepsilon}}) = \{\boldsymbol{\varepsilon} \mid \text{there is } \mathbf{u} \text{ with } \boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}\right] \text{in } \Omega, \quad \mathbf{u} = \bar{\boldsymbol{\varepsilon}}\mathbf{x} \quad \text{on } \partial\Omega\}, \tag{3}$$

where $\mathbf{u}$ is the displacement field and $\bar{\boldsymbol{\varepsilon}}$ is a constant second-order tensor. Note that, in this case, $\mathbf{u}$ is such that the average strain is simply $\langle \boldsymbol{\varepsilon} \rangle = \bar{\boldsymbol{\varepsilon}}$.

Alternatively, the behavior of the phases can be characterized by *stress potentials*, $u^{(r)}$, which are dual to the $w^{(r)}$, such that the local strain–stress relation is determined by:

$$\boldsymbol{\varepsilon} = \frac{\partial u^{(r)}}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}), \qquad u(\mathbf{x}, \boldsymbol{\sigma}) = \sum_{r=1}^{N} \chi^{(r)}(\mathbf{x}) u^{(r)}(\boldsymbol{\sigma}). \tag{4}$$

According to the minimum complementary energy principle, the *effective stress potential*, $\widetilde{U}$, can be written as:

$$\widetilde{U}(\bar{\boldsymbol{\sigma}}) = \min_{\boldsymbol{\sigma}\in\mathscr{S}(\bar{\boldsymbol{\sigma}})} \langle u(\mathbf{x},\boldsymbol{\sigma})\rangle = \min_{\boldsymbol{\sigma}\in\mathscr{S}(\bar{\boldsymbol{\sigma}})} \sum_{r=1}^{N} c^{(r)} \langle u^{(r)}(\boldsymbol{\sigma})\rangle^{(r)}, \tag{5}$$

where $\mathscr{S}(\bar{\boldsymbol{\sigma}})$ is the set of self-equilibrated stresses that are consistent with the average stress condition $\langle\boldsymbol{\sigma}\rangle = \bar{\boldsymbol{\sigma}}$.

It can be shown (see, for example, Ponte Castañeda and Suquet, 1998) that the average stress in the composite, $\bar{\boldsymbol{\sigma}}$, is related to the average strain, $\bar{\boldsymbol{\varepsilon}}$, through the relations:

$$\bar{\boldsymbol{\sigma}} = \frac{\partial\widetilde{W}}{\partial\bar{\boldsymbol{\varepsilon}}}(\bar{\boldsymbol{\varepsilon}}) \quad \text{and} \quad \bar{\boldsymbol{\varepsilon}} = \frac{\partial\widetilde{U}}{\partial\bar{\boldsymbol{\sigma}}}(\bar{\boldsymbol{\sigma}}), \tag{6}$$

which provide the macroscopic constitutive relation for the composite. Thus, if we know the effective energy functions, we can obtain the stress–strain relation for the nonlinear composite. Note that these functions are very difficult to compute, in general, since they correspond to the solution of a set of nonlinear partial differential equations with randomly oscillating coefficients. In the next section the new variational principles are used to generate estimates for $\widetilde{W}$ and $\widetilde{U}$.

3. Second-order homogenization estimates

In this section, an outline of the second-order homogenization method of Ponte Castañeda (2002a) is given. The idea is to construct a *linear comparison composite* whose effective potential can be used to estimate the effective potential of the *nonlinear composite*. The homogenization is thus carried out for a *linear* heterogeneous medium, for which many accurate estimates are already available (see, for example, Milton, 2002; Torquato, 2001). Let the comparison composite have a strain potential of the form:

$$w_{\mathrm{T}}(\mathbf{x},\boldsymbol{\varepsilon}) = \sum_{r=1}^{N} \chi^{(r)}(\mathbf{x}) w_{\mathrm{T}}^{(r)}(\boldsymbol{\varepsilon}), \tag{7}$$

where the $\chi^{(r)}$ are the same characteristic functions as the nonlinear composite's (i.e. both composites have the same microstructure), and the phase potentials $w_{\mathrm{T}}^{(r)}$ are second-order Taylor-type expressions:

$$w_{\mathrm{T}}^{(r)}(\boldsymbol{\varepsilon}) = w^{(r)}(\boldsymbol{\varepsilon}^{(r)}) + \frac{\partial w^{(r)}}{\partial\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}^{(r)})\cdot(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{(r)}) + \frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{(r)})\cdot\mathbf{L}_0^{(r)}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{(r)}). \tag{8}$$

In these last expressions, $\boldsymbol{\varepsilon}^{(r)}$ is a uniform reference strain, $\mathbf{L}_0^{(r)}$ is a symmetric, constant fourth-order tensor, and $w^{(r)}$ is the nonlinear potential of phase r. Differentiating this potential gives a stress–strain relation:

$$\boldsymbol{\sigma} = \frac{\partial w^{(r)}}{\partial\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}^{(r)}) + \mathbf{L}_0^{(r)}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{(r)}) = \boldsymbol{\tau}^{(r)} + \mathbf{L}_0^{(r)}\boldsymbol{\varepsilon}, \tag{9}$$

where the stress polarization tensors $\boldsymbol{\tau}^{(r)} = \partial w^{(r)}/\partial\boldsymbol{\varepsilon}(\boldsymbol{\varepsilon}^{(r)}) - \mathbf{L}_0^{(r)}\boldsymbol{\varepsilon}^{(r)}$ are mathematically equivalent to thermal stress tensors, since they are independent of the strain. Also note that $\mathbf{L}_0^{(r)}$ corresponds to the modulus tensor of the linear phase. The effective potential associated with the linear comparison composite with local potential given by (7) and (8) can be written as (Laws, 1973; Willis, 1981):

$$\widetilde{W}_{\mathrm{T}}(\bar{\boldsymbol{\varepsilon}}) = \min_{\boldsymbol{\varepsilon}\in\mathscr{K}(\bar{\boldsymbol{\varepsilon}})} \langle w_{\mathrm{T}}(\mathbf{x},\boldsymbol{\varepsilon})\rangle = \tilde{f} + \tilde{\boldsymbol{\tau}}\cdot\bar{\boldsymbol{\varepsilon}} + \frac{1}{2}\bar{\boldsymbol{\varepsilon}}\cdot\widetilde{\mathbf{L}}_0\bar{\boldsymbol{\varepsilon}}, \tag{10}$$

where $\tilde{f}$, $\tilde{\boldsymbol{\tau}}$ and $\widetilde{\mathbf{L}}_0$ are the relevant effective energy at zero applied strain, the effective polarization, and effective modulus tensor, respectively.

The idea of the second-order procedure is to choose, within certain simplifying assumptions, the reference strains and modulus tensors of the above-defined linear comparison composite, in such a way as to generate the best possible estimates for the nonlinear potential $\widetilde{W}$ through known estimates for the linear potential $\widetilde{W}_{\mathrm{T}}$. This optimization procedure, which involves some approximations, is not repeated here for brevity (see Ponte Castañeda (2002a) for details). In any event, the optimal values of the variables $\boldsymbol{\varepsilon}^{(r)}$ and $\mathbf{L}_0^{(r)}$ are given by:

$$\boldsymbol{\varepsilon}^{(r)} = \overline{\boldsymbol{\varepsilon}}^{(r)}, \tag{11a}$$

and

$$\frac{\partial w^{(r)}}{\partial \boldsymbol{\varepsilon}}(\hat{\boldsymbol{\varepsilon}}^{(r)}) - \frac{\partial w^{(r)}}{\partial \boldsymbol{\varepsilon}}(\overline{\boldsymbol{\varepsilon}}^{(r)}) = \mathbf{L}_0^{(r)}(\hat{\boldsymbol{\varepsilon}}^{(r)} - \overline{\boldsymbol{\varepsilon}}^{(r)}), \tag{11b}$$

where the $\hat{\boldsymbol{\varepsilon}}^{(r)}$ are constant second-order tensors, arising from the introduction of suitable error measures (Ponte Castañeda, 2002a), that depend on the second moments of the fluctuations of the strain through appropriate *traces* of the relations:

$$(\hat{\boldsymbol{\varepsilon}}^{(r)} - \overline{\boldsymbol{\varepsilon}}^{(r)}) \otimes (\hat{\boldsymbol{\varepsilon}}^{(r)} - \overline{\boldsymbol{\varepsilon}}^{(r)}) = \langle (\boldsymbol{\varepsilon} - \overline{\boldsymbol{\varepsilon}}^{(r)}) \otimes (\boldsymbol{\varepsilon} - \overline{\boldsymbol{\varepsilon}}^{(r)}) \rangle^{(r)} \doteq \mathbf{C}_{\boldsymbol{\varepsilon}}^{(r)}, \tag{12}$$

where $\mathbf{C}_{\boldsymbol{\varepsilon}}^{(r)}$ serves to denote the covariance tensor of the strain field in phase r of the linear comparison composite. It should be emphasized that, in general, the equality cannot be enforced for all components of the tensorial relation (12), and that is why only certain traces of this relation are used.

Thus, the reference strains $\boldsymbol{\varepsilon}^{(r)}$ are identified with the phase averages of the strain, and the modulus tensors $\mathbf{L}_0^{(r)}$ follow from the so-called "generalized secant condition" (11b). Fig. 1 shows a one-dimensional graphical representation of this condition. Note that, if the strain field in a phase is quite homogeneous, i.e. the fluctuations are small, the modulus tensor of the corresponding linearized phase in the comparison composite will be close to the tangent modulus tensor. But if the field has a heterogeneous distribution, the new method will result in a less stiff modulus tensor.

The average strains and the covariance tensors of the actual strain field may be computed "self-consistently" from the linear comparison composite using the following identities (Ponte Castañeda and Suquet, 1998):

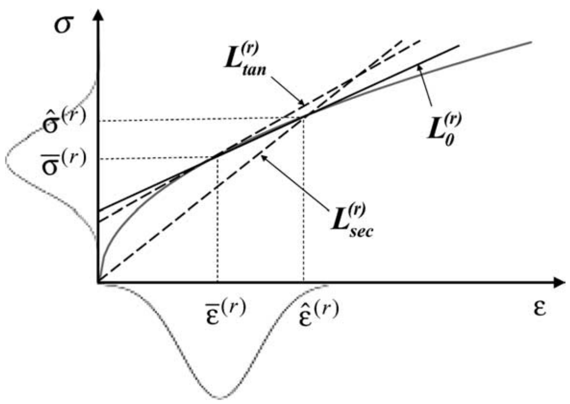

Fig. 1. One-dimensional sketch of the nonlinear stress–strain relation and different types of linearizations: $L_0^{(r)}$, $L_{\mathrm{sec}}^{(r)}$ and $L_{\mathrm{tan}}^{(r)}$ refer to the new "generalized secant", secant, and tangent modulus, respectively.

$$\bar{\boldsymbol{\varepsilon}}^{(r)} = \frac{1}{c^{(r)}}\frac{\partial(\widetilde{W}_{\rm T} - \bar{f})}{\partial \boldsymbol{\tau}^{(r)}}, \quad \text{and} \quad \mathbf{C}_{\boldsymbol{\varepsilon}}^{(r)} = \frac{2}{c^{(r)}}\frac{\partial \widetilde{W}_{\rm T}}{\partial \mathbf{L}_0^{(r)}}. \tag{13}$$

In the first relation,

$$\bar{f} = \sum_{r=1}^{N} c^{(r)} f^{(r)}, \qquad f^{(r)} = w^{(r)}(\bar{\boldsymbol{\varepsilon}}^{(r)}) - \boldsymbol{\tau}^{(r)}\cdot\bar{\boldsymbol{\varepsilon}}^{(r)} - \frac{1}{2}\bar{\boldsymbol{\varepsilon}}^{(r)}\cdot\mathbf{L}_0^{(r)}\bar{\boldsymbol{\varepsilon}}^{(r)}$$

and the $\mathbf{L}_0^{(r)}$ are held fixed, while in the second, the $\bar{\boldsymbol{\varepsilon}}^{(r)}$ are held fixed.

Finally, the effective potential of the nonlinear composite (2) may be re-expressed in terms of only the variables $\bar{\boldsymbol{\varepsilon}}^{(r)}$ and $\hat{\boldsymbol{\varepsilon}}^{(r)}$ via the relation (Ponte Castañeda, 2002a):

$$\widetilde{W}(\bar{\boldsymbol{\varepsilon}}) = \sum_{r=1}^{N} c^{(r)}\left[w^{(r)}(\hat{\boldsymbol{\varepsilon}}^{(r)}) - \frac{\partial w^{(r)}}{\partial \boldsymbol{\varepsilon}}(\bar{\boldsymbol{\varepsilon}}^{(r)})\cdot(\hat{\boldsymbol{\varepsilon}}^{(r)} - \bar{\boldsymbol{\varepsilon}}^{(r)})\right]. \tag{14}$$

Knowing the effective potential $\widetilde{W}_{\rm T}$ of the linear comparison composite as a function of the phase moduli $\mathbf{L}_0^{(r)}$, and the polarizations $\boldsymbol{\tau}^{(r)}$, the variables $\bar{\boldsymbol{\varepsilon}}^{(r)}$ and $\hat{\boldsymbol{\varepsilon}}^{(r)}$ can then be computed using (13), and an estimate for the nonlinear potential may be obtained via (14). Several methods are available to estimate and bound the effective potential of a linear composite, such as the Hashin–Shtrikman and self-consistent methods. If the method used is exact to second order in the heterogeneity contrast, it can be shown that the estimate (14) is also exact to second order, and therefore in agreement with the small-contrast expansion of Suquet and Ponte Castañeda (1993).

Next, consider the case of isotropic, incompressible phases with $w^{(r)}(\boldsymbol{\varepsilon}) = \phi^{(r)}(\varepsilon_{\rm e})$, where $\boldsymbol{\varepsilon}$ is assumed to be traceless, and $\varepsilon_{\rm e}$ is the von Mises equivalent strain, defined in terms of the strain deviatoric tensor $\boldsymbol{\varepsilon}_{\rm d}$ by $\varepsilon_{\rm e} = \sqrt{(2/3)\boldsymbol{\varepsilon}_{\rm d}\cdot\boldsymbol{\varepsilon}_{\rm d}}$. For this broad class of potentials, Ponte Castañeda (2002b) proposed, as an approximation, (incompressible) tensors $\mathbf{L}_0^{(r)}$ with principal axes aligned with the average strain, such that:

$$\mathbf{L}_0^{(r)} = 2\lambda_0^{(r)}\mathbf{E}^{(r)} + 2\mu_0^{(r)}\mathbf{F}^{(r)}. \tag{15}$$

Here, $\mathbf{E}^{(r)}$ and $\mathbf{F}^{(r)}$ are fourth-order projection tensors (Ponte Castañeda, 1996) defined by $\mathbf{E}^{(r)} = (2/3)\check{\boldsymbol{\varepsilon}}_{\rm d}^{(r)}\otimes\check{\boldsymbol{\varepsilon}}_{\rm d}^{(r)}$, $\mathbf{E}^{(r)} + \mathbf{F}^{(r)} = \mathbf{K}$, with $\check{\boldsymbol{\varepsilon}}_{\rm d}^{(r)} = \bar{\boldsymbol{\varepsilon}}^{(r)}/\bar{\varepsilon}_{\rm e}^{(r)}$, such that $\mathbf{E}^{(r)}\mathbf{E}^{(r)} = \mathbf{E}^{(r)}$, $\mathbf{F}^{(r)}\mathbf{F}^{(r)} = \mathbf{F}^{(r)}$, $\mathbf{E}^{(r)}\mathbf{F}^{(r)} = \mathbf{F}^{(r)}\mathbf{E}^{(r)} = \mathbf{0}$. Also, $\mathbf{K}$ is the standard fourth-order isotropic shear projection tensor. Note that although the nonlinear phases are isotropic, the phases of the linear comparison composite are anisotropic. With this choice of $\mathbf{L}_0^{(r)}$, it follows from (11b) that the traceless tensors $\hat{\boldsymbol{\varepsilon}}^{(r)}$ have components "parallel" and "perpendicular" to the average fields, which, from (12), are given by:

$$\hat{\varepsilon}_{\parallel}^{(r)} = \bar{\varepsilon}_{\rm e}^{(r)} + \sqrt{\tfrac{2}{3}\mathbf{E}^{(r)}\cdot\mathbf{C}_{\boldsymbol{\varepsilon}}^{(r)}}, \qquad \hat{\varepsilon}_{\perp}^{(r)} = \sqrt{\tfrac{2}{3}\mathbf{F}^{(r)}\cdot\mathbf{C}_{\boldsymbol{\varepsilon}}^{(r)}}, \tag{16}$$

where $\hat{\varepsilon}_{\parallel}^{(r)} = \left(\frac{2}{3}\hat{\boldsymbol{\varepsilon}}^{(r)}\cdot\mathbf{E}^{(r)}\hat{\boldsymbol{\varepsilon}}^{(r)}\right)^{1/2}$ and $\hat{\varepsilon}_{\perp}^{(r)} = \left(\frac{2}{3}\hat{\boldsymbol{\varepsilon}}^{(r)}\cdot\mathbf{F}^{(r)}\hat{\boldsymbol{\varepsilon}}^{(r)}\right)^{1/2}$, so that $(\hat{\varepsilon}_{\rm e}^{(r)})^2 = (\hat{\varepsilon}_{\parallel}^{(r)})^2 + (\hat{\varepsilon}_{\perp}^{(r)})^2$. The "generalized secant" conditions (11b) reduce to:

$$3\lambda_0^{(r)}\left(\hat{\varepsilon}_{\parallel}^{(r)} - \bar{\varepsilon}_{\rm e}^{(r)}\right) = \phi^{(r)'}(\hat{\varepsilon}_{\rm e}^{(r)})\frac{\hat{\varepsilon}_{\parallel}^{(r)}}{\hat{\varepsilon}_{\rm e}^{(r)}} - \phi^{(r)'}\left(\bar{\varepsilon}_{\rm e}^{(r)}\right), \qquad 3\mu_0^{(r)} = \frac{\phi^{(r)'}(\hat{\varepsilon}_{\rm e}^{(r)})}{\hat{\varepsilon}_{\rm e}^{(r)}}. \tag{17}$$

Finally, expression (14) for $\widetilde{W}$ simplifies to:

$$\widetilde{W}(\bar{\boldsymbol{\varepsilon}}) = \sum_{r=1}^{N} c^{(r)}\left[\phi^{(r)}(\hat{\varepsilon}_{\rm e}^{(r)}) - \phi^{(r)'}\left(\bar{\varepsilon}_{\rm e}^{(r)}\right)\left(\hat{\varepsilon}_{\parallel}^{(r)} - \bar{\varepsilon}_{\rm e}^{(r)}\right)\right]. \tag{18}$$

Proceeding in a completely analogous fashion, estimates for $\widetilde{U}$ can be obtained using the stress potentials $u^{(r)}$ and their corresponding second-order Taylor-type expressions $u_{\rm T}^{(r)}$:

$$\widetilde{U}(\bar{\boldsymbol{\sigma}}) = \sum_{r=1}^{N} c^{(r)} \left[u^{(r)}(\hat{\boldsymbol{\sigma}}^{(r)}) - \frac{\partial u^{(r)}}{\partial \boldsymbol{\sigma}}(\bar{\boldsymbol{\sigma}}^{(r)}) \cdot (\hat{\boldsymbol{\sigma}}^{(r)} - \bar{\boldsymbol{\sigma}}^{(r)}) \right], \tag{19}$$

where the uniform reference stresses $\boldsymbol{\sigma}^{(r)}$ have been identified with the average stresses in each phase $\bar{\boldsymbol{\sigma}}^{(r)}$, and $\hat{\boldsymbol{\sigma}}^{(r)}$ are constant tensors that depend on the stress fluctuations through appropriate *traces* of the relations:

$$(\hat{\boldsymbol{\sigma}}^{(r)} - \bar{\boldsymbol{\sigma}}^{(r)}) \otimes (\hat{\boldsymbol{\sigma}}^{(r)} - \bar{\boldsymbol{\sigma}}^{(r)}) = \langle (\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}^{(r)}) \otimes (\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}^{(r)}) \rangle^{(r)} \doteq \mathbf{C}_{\boldsymbol{\sigma}}^{(r)}, \tag{20}$$

where $\mathbf{C}_{\boldsymbol{\sigma}}^{(r)}$ is the covariance tensor of the stress field in phase r. The phases of the linear thermoelastic comparison composite have strain polarizations $\boldsymbol{\eta}^{(r)} = \partial u^{(r)}/\partial \boldsymbol{\sigma}(\bar{\boldsymbol{\sigma}}^{(r)}) - \mathbf{M}_0^{(r)}\bar{\boldsymbol{\sigma}}^{(r)}$, and compliances $\mathbf{M}_0^{(r)}$ given by the secant-type condition:

$$\frac{\partial u^{(r)}}{\partial \boldsymbol{\sigma}}(\hat{\boldsymbol{\sigma}}^{(r)}) - \frac{\partial u^{(r)}}{\partial \boldsymbol{\sigma}}(\bar{\boldsymbol{\sigma}}^{(r)}) = \mathbf{M}_0^{(r)}(\hat{\boldsymbol{\sigma}}^{(r)} - \bar{\boldsymbol{\sigma}}^{(r)}). \tag{21}$$

Again, consider the case of isotropic, incompressible phases with $u^{(r)}(\boldsymbol{\sigma}) = \psi^{(r)}(\sigma_e)$, where σ_e is the von Mises equivalent stress, defined in terms of the stress deviatoric tensor $\boldsymbol{\sigma}_d$ by $\sigma_e = \sqrt{(2/3)\boldsymbol{\sigma}_d \cdot \boldsymbol{\sigma}_d}$. As in the strain formulation, for this class of potentials, we restrict attention to (incompressible) compliance tensors whose principal axes are aligned with the average stress:

$$\mathbf{M}_0^{(r)} = \frac{1}{2\lambda_0^{(r)}}\mathbf{E}^{(r)} + \frac{1}{2\mu_0^{(r)}}\mathbf{F}^{(r)}, \tag{22}$$

where $\mathbf{E}^{(r)} = (3/2)\check{\boldsymbol{\sigma}}_d^{(r)} \otimes \check{\boldsymbol{\sigma}}_d^{(r)}$, $\mathbf{E}^{(r)} + \mathbf{F}^{(r)} = \mathbf{K}$, with $\check{\boldsymbol{\sigma}}_d^{(r)} = \bar{\boldsymbol{\sigma}}^{(r)}/\bar{\sigma}_e^{(r)}$, are the appropriate projection tensors in this case. From (20) and (21) it follows that:

$$\hat{\sigma}_{\parallel}^{(r)} = \bar{\sigma}_e^{(r)} + \sqrt{\tfrac{3}{2}\mathbf{E}^{(r)} \cdot \mathbf{C}_{\boldsymbol{\sigma}}^{(r)}}, \qquad \hat{\sigma}_{\perp}^{(r)} = \sqrt{\tfrac{3}{2}\mathbf{F}^{(r)} \cdot \mathbf{C}_{\boldsymbol{\sigma}}^{(r)}}, \tag{23}$$

where $\hat{\sigma}_{\parallel}^{(r)} = \left(\tfrac{3}{2}\hat{\boldsymbol{\sigma}}^{(r)} \cdot \mathbf{E}^{(r)}\hat{\boldsymbol{\sigma}}^{(r)}\right)^{1/2}$ and $\hat{\sigma}_{\perp}^{(r)} = \left(\tfrac{3}{2}\hat{\boldsymbol{\sigma}}^{(r)} \cdot \mathbf{F}^{(r)}\hat{\boldsymbol{\sigma}}^{(r)}\right)^{1/2}$, are the "parallel" and "perpendicular" components of the traceless tensors $\hat{\boldsymbol{\sigma}}^{(r)}$, respectively. The "generalized secant conditions" (21) reduce to:

$$\frac{1}{3\lambda_0^{(r)}}\left(\hat{\sigma}_{\parallel}^{(r)} - \bar{\sigma}_e^{(r)}\right) = \psi^{(r)'}(\hat{\sigma}_e^{(r)})\frac{\hat{\sigma}_{\parallel}^{(r)}}{\hat{\sigma}_e^{(r)}} - \psi^{(r)'}\left(\bar{\sigma}_e^{(r)}\right), \qquad \frac{1}{3\mu_0^{(r)}} = \frac{\psi^{(r)'}(\hat{\sigma}_e^{(r)})}{\hat{\sigma}_e^{(r)}}. \tag{24}$$

Finally, expression (19) can be written as:

$$\widetilde{U}(\bar{\boldsymbol{\sigma}}) = \sum_{r=1}^{N} c^{(r)}\left[\psi^{(r)}(\hat{\sigma}_e^{(r)}) - \psi^{(r)'}\left(\bar{\sigma}_e^{(r)}\right)\left(\hat{\sigma}_{\parallel}^{(r)} - \bar{\sigma}_e^{(r)}\right)\right]. \tag{25}$$

Relations (14) and (19) provide two different ways to estimate the effective behavior of the nonlinear composites. However, it is important to emphasize that, because of the approximations introduced in the optimization procedure, these estimates are not exactly equivalent (see Ponte Castañeda, 2002a), and a small *duality gap* is expected, in general.

A *third* way to approximate the constitutive behavior of the nonlinear composite is to use, directly, the constitutive relations of the associated linear comparison composite, as given by (Laws, 1973):

$$\bar{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\tau}} + \widetilde{\mathbf{L}}_0\bar{\boldsymbol{\varepsilon}}, \qquad \bar{\boldsymbol{\varepsilon}} = \tilde{\boldsymbol{\eta}} + \widetilde{\mathbf{M}}_0\bar{\boldsymbol{\sigma}}. \tag{26}$$

Making use of well-known expressions for the modulus and compliance tensors $\widetilde{\mathbf{L}}_0$ and $\widetilde{\mathbf{M}}_0$, and the stress and strain polarizations $\tilde{\boldsymbol{\tau}}$ and $\tilde{\boldsymbol{\eta}}$, together with the expressions for the phase polarizations $\boldsymbol{\tau}^{(r)}$ and $\boldsymbol{\eta}^{(r)}$ in terms of the phase averages $\bar{\boldsymbol{\varepsilon}}^{(r)}$ and $\bar{\boldsymbol{\sigma}}^{(r)}$, and modulus tensors $\mathbf{L}_0^{(r)}$ and $\mathbf{M}_0^{(r)}$, these expressions may be re-written more explicitly as (Ponte Castañeda, 2002a):

$$\bar{\boldsymbol{\sigma}} = \sum_{r=1}^{N} c^{(r)} \frac{\partial w^{(r)}}{\partial \boldsymbol{\varepsilon}}(\bar{\boldsymbol{\varepsilon}}^{(r)}), \quad \text{and} \quad \bar{\boldsymbol{\varepsilon}} = \sum_{r=1}^{N} c^{(r)} \frac{\partial u^{(r)}}{\partial \boldsymbol{\sigma}}(\bar{\boldsymbol{\sigma}}^{(r)}). \tag{27}$$

These two stress–strain relations for the nonlinear composite are exactly equivalent to each other, because there is *no* duality gap for the linear comparison composite. However, again for reasons related to the approximations mentioned above, they can be shown to be *different* from the corresponding relations for the nonlinear composite generated by direct derivation (6) of the second-order estimates (14) and (19). They can be thought of as improved versions of the "affine" estimates of Masson et al. (2000), in the same sense as the second-order estimates of Ponte Castañeda (2002a) are improved versions of the earlier second-order estimates of Ponte Castañeda (1996). Unfortunately, these new "affine" estimates are not exact to second-order in the contrast, and are expected to be less accurate than the corresponding estimates (14) and (19).

Finally, since the linear phase potentials $w_{\mathrm{T}}^{(r)}$ and $u_{\mathrm{T}}^{(r)}$ are dual to each other, it is worth noting that the following duality relations hold between the strain/moduli variables in (14) and stress/compliance variables in (19):

$$\begin{aligned}
&\bar{\boldsymbol{\sigma}}^{(r)} = \frac{\partial w^{(r)}}{\partial \boldsymbol{\varepsilon}}(\bar{\boldsymbol{\varepsilon}}^{(r)}), \qquad \bar{\boldsymbol{\varepsilon}}^{(r)} = \frac{\partial u^{(r)}}{\partial \boldsymbol{\sigma}}(\bar{\boldsymbol{\sigma}}^{(r)}),\\
&\hat{\boldsymbol{\sigma}}^{(r)} = \frac{\partial w^{(r)}}{\partial \boldsymbol{\varepsilon}}(\hat{\boldsymbol{\varepsilon}}^{(r)}), \qquad \hat{\boldsymbol{\varepsilon}}^{(r)} = \frac{\partial u^{(r)}}{\partial \boldsymbol{\sigma}}(\hat{\boldsymbol{\sigma}}^{(r)}),\\
&\mathbf{M}_0^{(r)} = \left(\mathbf{L}_0^{(r)}\right)^{-1},
\end{aligned} \tag{28}$$

provided $\bar{\boldsymbol{\varepsilon}}$ and $\bar{\boldsymbol{\sigma}}$ are taken to be related by expressions (26), or equivalently, by expressions (27).

4. Two-phase, power-law fibrous composites under anti-plane or in-plane loading

In this section we consider fibrous composites with incompressible power-law phases subject to anti-plane or in-plane loading. The phase strain and stress potentials are given by:

$$w^{(r)}(\boldsymbol{\varepsilon}) = \frac{\varepsilon_0 \sigma_0^{(r)}}{1+m}\left(\frac{\varepsilon_{\mathrm{e}}}{\varepsilon_0}\right)^{1+m}, \tag{29a}$$

$$u^{(r)}(\boldsymbol{\sigma}) = \frac{\varepsilon_0 \sigma_0^{(r)}}{1+n}\left(\frac{\sigma_{\mathrm{e}}}{\sigma_0^{(r)}}\right)^{1+n}, \tag{29b}$$

respectively. In these expressions, m is the strain-hardening parameter, such that $0 \leqslant m \leqslant 1$, $n = 1/m$ is the corresponding nonlinearity exponent, $\sigma_0^{(r)}$ is the flow stress of phase r, ε_0 is a reference strain, and the ε_{e} and σ_{e} are the von Mises equivalent strain and stress, already introduced in the previous section. The stress–strain relation for such a material is given by:

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}) = -p\mathbf{I} + \frac{2}{3}\frac{\sigma_0}{\varepsilon_0}\left(\frac{\varepsilon_{\mathrm{e}}}{\varepsilon_0}\right)^{m-1}\boldsymbol{\varepsilon}_{\mathrm{d}}, \tag{30}$$

where $p = -\mathrm{tr}(\boldsymbol{\sigma})/3$ is the indeterminate, hydrostatic stress associated with the incompressibility condition $\mathrm{tr}(\boldsymbol{\varepsilon}) = 0$. Note that $m = 1$ and $m = 0$ represent linear and rigid-perfectly plastic behavior, respectively. This model is commonly used to characterize time-independent plastic deformation of metals, as well as their time-dependent viscous deformation (e.g. high temperature creep). In the first case, the deformations are infinitesimal and $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ represent the infinitesimal stress and strain tensors, respectively. In the second case, the deformations are finite and $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are identified with the Cauchy stress and Eulerian strain-rate, respectively. Then, m becomes a strain-rate sensitivity parameter. Although we will continue to use only infinitesimal stresses and strains below, reference will also be made to the rate-sensitive case, without further clarification.

The infinitely long fibers are assumed to be aligned and perfectly bonded to the matrix, and to have circular cross section with diameter much smaller than the dimensions of the specimen. The distribution of the fibers in the transverse plane is assumed *random* and *isotropic*, so the composite is transversely isotropic. Furthermore, from the homogeneity of the potentials (29a) and (29b) in their corresponding fields, it follows that a transversely isotropic composite, made up of power-law phases with the same exponent m and the same reference strain ε_0, subject to anti-plane or in-plane loading, has effective potentials of the form (29a) and (29b). They can be written as:

$$\widetilde{W}(\bar{\boldsymbol{\varepsilon}}) = \frac{\varepsilon_0 \tilde{\sigma}_0}{1+m}\left(\frac{\bar{\varepsilon}_e}{\varepsilon_0}\right)^{1+m}, \tag{31a}$$

$$\widetilde{U}(\bar{\boldsymbol{\sigma}}) = \frac{\varepsilon_0 \tilde{\sigma}_0}{1+n}\left(\frac{\bar{\sigma}_e}{\tilde{\sigma}_0}\right)^{1+n}, \tag{31b}$$

where $\bar{\varepsilon}_e$ and $\bar{\sigma}_e$ are the equivalent average strain and stress, respectively. For anti-plane loading along the 3-direction they reduce to $\bar{\varepsilon}_e = (2/\sqrt{3})\sqrt{\bar{\varepsilon}_{13}^2 + \bar{\varepsilon}_{23}^2}$ and $\bar{\sigma}_e = (\sqrt{3})\sqrt{\bar{\sigma}_{13}^2 + \bar{\sigma}_{23}^2}$, and for in-plane loading they reduce to $\bar{\varepsilon}_e = (2/\sqrt{3})\sqrt{\bar{\varepsilon}_{12}^2 + \frac{1}{4}(\bar{\varepsilon}_{11} - \bar{\varepsilon}_{22})^2}$ and $\bar{\sigma}_e = (\sqrt{3})\sqrt{\bar{\sigma}_{12}^2 + \frac{1}{4}(\bar{\sigma}_{11} - \bar{\sigma}_{22})^2}$. The effective flow stress $\tilde{\sigma}_0$ is a function of the nonlinearity, the contrast, and concentration of fibers, and it completely characterizes the effective behavior.

Before proceeding to the computation of the effective potentials (31a) and (31b) for the fibrous composites, we note that the effective energy (10) of the N-phase thermoelastic comparison composite simplifies greatly when the composite has only two-phases. In this case, the Levin relations (Levin, 1967) can be used to obtain the effective thermal stress tensor in terms of the effective elastic tensor. The effective energy then takes the form:

$$\widetilde{W}_T(\bar{\boldsymbol{\varepsilon}}) = \bar{f} + \bar{\boldsymbol{\tau}}\cdot\bar{\boldsymbol{\varepsilon}} + \tfrac{1}{2}\bar{\boldsymbol{\varepsilon}}\cdot\bar{\mathbf{L}}_0\bar{\boldsymbol{\varepsilon}} + \tfrac{1}{2}\left[\bar{\boldsymbol{\varepsilon}} + (\Delta\mathbf{L}_0)^{-1}(\Delta\boldsymbol{\tau})\right]\cdot(\widetilde{\mathbf{L}}_0 - \bar{\mathbf{L}}_0)\left[\bar{\boldsymbol{\varepsilon}} + (\Delta\mathbf{L}_0)^{-1}(\Delta\boldsymbol{\tau})\right], \tag{32}$$

where the overbar denotes volume averages, $\Delta\mathbf{L}_0 = \mathbf{L}_0^{(1)} - \mathbf{L}_0^{(2)}$ and $\Delta\boldsymbol{\tau} = \boldsymbol{\tau}^{(1)} - \boldsymbol{\tau}^{(2)}$. Note that the only nonexplicit term in this expression is the tensor of effective moduli $\widetilde{\mathbf{L}}_0$ for a two-phase, linear-elastic composite. Estimates of the Hashin and Shtrikman (1963) type for such linear composites with particulate-type microstructures (i.e., inclusions of phase 2 dispersed in a matrix of phase 1) have been given by Willis (1977, 1978) and Ponte Castañeda and Willis (1995). The relevant expression for the effective modulus tensor is:

$$\widetilde{\mathbf{L}}_0 = \sum_{r=1}^{2} c^{(r)}\mathbf{L}_0^{(r)}\left[\mathbf{I} + \mathbf{P}^{(0)}(\mathbf{L}_0^{(r)} - \mathbf{L}^{(0)})\right]^{-1}\left\{\sum_{s=1}^{2} c^{(s)}\left[\mathbf{I} + \mathbf{P}^{(0)}(\mathbf{L}_0^{(s)} - \mathbf{L}^{(0)})\right]^{-1}\right\}^{-1}, \tag{33}$$

where the modulus tensor $\mathbf{L}^{(0)}$ of the homogeneous reference medium in the Hashin–Shtrikman approximation must be identified with the modulus tensor of the matrix phase ($\mathbf{L}_0^{(1)}$, in this case), and $\mathbf{P}^{(0)}$ is a microstructural tensor, related to the Eshelby tensor, which depends on $\mathbf{L}^{(0)}$, the shape and orientation of the particles, as well as on the shape and orientation of the two-point correlation functions for their distribution in space. These estimates are known to be exact to first order in the volume fraction of the particles and to second order in the heterogeneity contrast. They tend to underestimate the interaction between particles, but can give fairly accurate estimates for small to intermediate concentrations.

Since the nonlinear phases are isotropic, and are isotropically distributed in the transverse plane, under the assumptions of anti-plane or in-plane strain loading, it is reasonable to assume that the average strain field in the phases is aligned with the average strain, i.e. $\check{\boldsymbol{\varepsilon}}_{\rm d}^{(r)} = \check{\boldsymbol{\varepsilon}} = \bar{\boldsymbol{\varepsilon}}/\bar{\varepsilon}_{\rm e}$ for all r, such that the phase projection tensors become identical for both phases, and are given by $\mathbf{E} = (2/3)\check{\boldsymbol{\varepsilon}} \otimes \check{\boldsymbol{\varepsilon}}$ and $\mathbf{F} = \mathbf{K} - \mathbf{E}$. Then, using the fact that $\mathbf{L}^{(0)}$ has the form (15), and making use of the long-fiber limit in the appropriate expressions for the tensor $\mathbf{P}^{(0)}$, it can be shown (see Ponte Castañeda, 1996) that under in-plane and anti-plane loading, the in-plane and anti-plane components of the tensor $\mathbf{P}^{(0)}$, respectively, may be written in the form:

$$\mathbf{P}^{(0)} = \frac{\sqrt{k}}{2(1+\sqrt{k})\lambda^{(0)}}\mathbf{E} + \frac{1}{2(1+\sqrt{k})\mu^{(0)}}\mathbf{F}, \tag{34}$$

where $k = \lambda^{(0)}/\mu^{(0)}$ is the anisotropy ratio of the homogeneous reference medium, and the projection tensors have to be suitably interpreted.

With expressions (32)–(34) defining explicitly the effective energy of the relevant linear comparison composite, we have everything required to compute the effective energies of the power-law fibrous composites. Thus, introducing (29a) and (31a) into (18), we arrive at the following expression for the normalized effective flow stress:

$$\frac{\tilde{\sigma}_0}{\sigma_0^{(1)}} = c^{(1)}\left[\left(\frac{\hat{\varepsilon}_{\rm e}^{(1)}}{\bar{\varepsilon}_{\rm e}}\right)^{1+m} - (1+m)\left(\frac{\bar{\varepsilon}_{\rm e}^{(1)}}{\bar{\varepsilon}_{\rm e}}\right)^{m}\left(\frac{\hat{\varepsilon}_{\parallel}^{(1)}}{\bar{\varepsilon}_{\rm e}} - \frac{\bar{\varepsilon}_{\rm e}^{(1)}}{\bar{\varepsilon}_{\rm e}}\right)\right] + c^{(2)}\frac{\sigma_0^{(2)}}{\sigma_0^{(1)}}\left(\frac{\bar{\varepsilon}_{\rm e}^{(2)}}{\bar{\varepsilon}_{\rm e}}\right)^{1+m}, \tag{35}$$

where it is recalled that the labels 1 and 2 have been used to identify the matrix and fiber phases, respectively. Note that $\bar{\varepsilon}_{\rm e}^{(2)}$ can be eliminated in favor of $\bar{\varepsilon}_{\rm e}^{(1)}$ using the average strain condition, i.e. $\bar{\varepsilon}_{\rm e}^{(2)} = (\bar{\varepsilon}_{\rm e} - c^{(1)}\bar{\varepsilon}_{\rm e}^{(1)})/c^{(2)}$, and that the variables $\hat{\varepsilon}_{\parallel}^{(2)}$ and $\hat{\varepsilon}_{\perp}^{(2)}$ do not appear in (35) because there are no fluctuations in phase 2. This last result is associated with the Hashin–Shtrikman approximation and can be verified by noting that the tensor $\mathbf{P}^{(0)}$ is independent of $\mathbf{L}_0^{(2)}$ in this case. Expression (35) allows the computation of $\tilde{\sigma}_0$ as a function of the rate-sensitivity m, the fiber concentration $c^{(2)}$, and the contrast $\sigma_0^{(2)}/\sigma_0^{(1)}$, in terms of the variables $\bar{\varepsilon}_{\rm e}^{(1)}$, $\hat{\varepsilon}_{\parallel}^{(1)}$ and $\hat{\varepsilon}_{\perp}^{(1)}$, which, in turn, must be determined from a set of three algebraic nonlinear equations in these unknowns arising from expressions (13), together with (32)–(34), as well as relations (17) and (16).

The stress potential (29b) can be used as the starting point to generate alternative estimates for $\tilde{\sigma}_0$. In this case, the effective stress potential of the linear comparison composite is given in terms of the compliances and strain polarizations by an expression analogous to (32). In turn, the effective compliance tensor $\widetilde{\mathbf{M}}_0$ is given in terms of the compliances by an expression analogous to (33), where the relevant microstructural tensor is related to the $\mathbf{P}$-tensor (34) by $\mathbf{Q}^{(0)} = \left(\mathbf{M}^{(0)}\right)^{-1} - \left(\mathbf{M}^{(0)}\right)^{-1}\mathbf{P}^{(0)}\left(\mathbf{M}^{(0)}\right)^{-1}$. From (25), the normalized effective flow stress may be expressed in terms of the stress variables via:

$$\frac{\tilde{\sigma}_0}{\sigma_0^{(1)}} = \left\{c^{(1)}\left[\left(\frac{\hat{\sigma}_{\rm e}^{(1)}}{\bar{\sigma}_{\rm e}}\right)^{1+n} - (1+n)\left(\frac{\bar{\sigma}_{\rm e}^{(1)}}{\bar{\sigma}_{\rm e}}\right)^{n}\left(\frac{\hat{\sigma}_{\parallel}^{(1)}}{\bar{\sigma}_{\rm e}} - \frac{\bar{\sigma}_{\rm e}^{(1)}}{\bar{\sigma}_{\rm e}}\right)\right] + c^{(2)}\left(\frac{\sigma_0^{(2)}}{\sigma_0^{(1)}}\right)^{-n}\left(\frac{\bar{\sigma}_{\rm e}^{(2)}}{\bar{\sigma}_{\rm e}}\right)^{1+n}\right\}^{-1/n}, \tag{36}$$

where the variables $\bar{\sigma}_{\rm e}^{(1)}$, $\hat{\sigma}_{\parallel}^{(1)}$, and $\hat{\sigma}_{\perp}^{(1)}$ may be obtained from expressions completely analogous to the above-mentioned expressions in the context of the variables $\bar{\varepsilon}_{\rm e}^{(1)}$, $\hat{\varepsilon}_{\parallel}^{(1)}$ and $\hat{\varepsilon}_{\perp}^{(1)}$. However, they may also be computed with the help of the duality relations (28).

Finally, a third expression for $\tilde{\sigma}_0$ is obtained by making use of the affine version of the estimates, as specified by relations (27). For example, the first of them gives the expression:

$$\frac{\tilde{\sigma}_0}{\sigma_0^{(1)}} = c^{(1)}\left(\frac{\bar{\varepsilon}_{\rm e}^{(1)}}{\bar{\varepsilon}_{\rm e}}\right)^{m} + c^{(2)}\frac{\sigma_0^{(2)}}{\sigma_0^{(1)}}\left(\frac{\bar{\varepsilon}_{\rm e}^{(2)}}{\bar{\varepsilon}_{\rm e}}\right)^{m}. \tag{37}$$

Expressions (35)–(37) generalize corresponding expressions for the extreme cases of rigid particles and voids (infinite contrast) given by Ponte Castañeda (2002b). This author also gave estimates of the self-consistent type for these special case, where the fluctuations are nonzero in both phases in general. As a consequence of the duality gap, these three expressions are expected to give slightly different predictions for $\tilde{\sigma}_0$ for any m different from 1 (the linear limit). However, as verified in the next section, these expressions all agree in the ideally plastic (rate-insensitive) limit.

It is emphasized that because of the above-stated reasons relating to expression (34), all the results presented below, with the appropriate interpretations, are valid for both anti-plane and in-plane loading of the two-phase fibrous composites, even if the stress and strain fields would obviously differ in detail due to the geometric differences between anti-plane and in-plane loading conditions.

4.1. Hashin–Shtrikman estimates for rigid-perfectly plastic phases

The above expressions simplify considerably for the special case of rigid-perfectly plastic behavior, which corresponds to setting $m = 0$ in potentials (29a) and (29b). However when taking the limit $m \to 0$, we must consider two cases separately.

If the fibers are *stronger* than the matrix ($\sigma_0^{(2)}/\sigma_0^{(1)} \geqslant 1$) the solution can be shown to reduce to that for rigid particles, regardless of the contrast. In this case, the average strain in the particles is zero, and in the matrix we have $\bar{\varepsilon}_e^{(1)} = \bar{\varepsilon}_e/c^{(1)}$, $\hat{\varepsilon}_\parallel^{(1)} \to \infty$, and $\hat{\varepsilon}_\perp^{(1)} = 0$. The average stress in the matrix is the flow stress, i.e. $\bar{\sigma}_e^{(1)} = \sigma_0^{(1)}$, and the stress fluctuations are such that $\hat{\sigma}_\parallel^{(1)} = \sigma_0^{(1)}$ and $\hat{\sigma}_\perp^{(1)} = 0$, respectively. All three of the above expressions for the effective flow stress, (35)–(37), reduce to the result:

$$\tilde{\sigma}_0/\sigma_0^{(1)} = 1. \tag{38}$$

That is, there is no reinforcement effect by the harder fibers in this case.

If the fibers are *weaker* than the matrix ($\sigma_0^{(2)}/\sigma_0^{(1)} < 1$), it is important to realize that when $m \to 0$, the average strain in the matrix goes to zero exponentially, $\bar{\varepsilon}_e^{(1)} \sim e^{-\alpha(k)/m}$, in such a way that the average stress in this phase, which is proportional to $\left(\bar{\varepsilon}_e^{(1)}\right)^m$, is finite and below $\sigma_0^{(1)}$. Thus, in the matrix $\bar{\varepsilon}_e^{(1)} = 0$, so that $\bar{\varepsilon}_e^{(2)} = \bar{\varepsilon}_e/c^{(2)}$, and from relations (16):

$$\frac{\hat{\varepsilon}_\parallel^{(1)}}{\bar{\varepsilon}_e} = \frac{1}{\sqrt{2c^{(2)}}}\frac{1}{k^{1/4}}, \qquad \frac{\hat{\varepsilon}_\perp^{(1)}}{\bar{\varepsilon}_e} = \frac{1}{\sqrt{2c^{(2)}}}k^{1/4}, \tag{39}$$

where the anisotropy ratio k is determined as a function of the contrast and concentration from:

$$\frac{k^{3/4}}{1-k} = \sqrt{\frac{c^{(2)}}{2}}\left(1 - \frac{\sigma_0^{(2)}}{\sigma_0^{(1)}}\frac{\sqrt{1+k}}{1-k}\right), \tag{40}$$

which follows from the generalized secant condition (17) in phase 1.

The corresponding phase average and fluctuations of the stress can be deduced from (28). They are given in terms of the anisotropy ratio by:

$$\frac{\bar{\sigma}_e^{(1)}}{\sigma_0^{(1)}} = \frac{1-k}{\sqrt{1+k}}, \qquad \frac{\hat{\sigma}_\parallel^{(1)}}{\sigma_0^{(1)}} = \frac{1}{\sqrt{1+k}}, \qquad \frac{\hat{\sigma}_\perp^{(1)}}{\sigma_0^{(1)}} = \frac{\sqrt{k}}{\sqrt{1+k}}. \tag{41}$$

Finally, expressions (35)–(37) for the normalized effective flow stress all simplify to:

$$\frac{\tilde{\sigma}_0}{\sigma_0^{(1)}} = c^{(2)}\frac{\sigma_0^{(2)}}{\sigma_0^{(1)}} + (1 - c^{(2)})\frac{1-k}{\sqrt{1+k}}. \tag{42}$$

When $\sigma_0^{(2)}/\sigma_0^{(1)} \to 0$, these expressions reduce to the results of Ponte Castañeda (2002b) for the special case of aligned cylindrical voids distributed isotropically in a rigid-perfectly plastic matrix with zero hydrostatic strain.

4.2. Small contrast expansions

As already mentioned, estimates (35) and (36) are exact to second order in the heterogeneity contrast, that is, they both agree with the exact second-order asymptotic expansion of Ponte Castañeda and Suquet (1995), which for this case can be written as:

$$\tilde{\sigma}_0 = \langle \sigma_0 \rangle - \frac{1}{2} \frac{1+m}{\sqrt{m}+m} \frac{\langle \sigma_0^2 \rangle - \langle \sigma_0 \rangle^2}{\langle \sigma_0 \rangle}. \tag{43}$$

The first term in this expansion corresponds to the Voigt upper bound. Note that the range of validity of (43) vanishes as $m \to 0$. In fact, the estimate for the rigid-perfectly plastic limit ($m = 0$) has an expansion of a different form, which actually depends on whether the fibers are stronger or weaker than the matrix. Thus, for $\sigma_0^{(2)}/\sigma_0^{(1)} \geqslant 1$ the result is independent of the contrast, i.e. $\tilde{\sigma}_0 = \sigma_0^{(1)}$, while for $\sigma_0^{(2)}/\sigma_0^{(1)} < 1$ it is given by:

$$\tilde{\sigma}_0 = \langle \sigma_0 \rangle - \frac{3}{2}\left(1 - c^{(2)}\right)^{1/3} \left(\frac{\langle \sigma_0^2 \rangle - \langle \sigma_0 \rangle^2}{\langle \sigma_0 \rangle} \right)^{2/3}, \tag{44}$$

which is the small-contrast expansion of expression (42).

On the other hand, the affine estimate (37), which is known not to be exact to second order in the contrast, has an expansion of the form:

$$\tilde{\sigma}_0 = \langle \sigma_0 \rangle - \frac{1}{2} \frac{\langle \sigma_0^2 \rangle - \langle \sigma_0 \rangle^2}{\langle \sigma_0 \rangle}, \tag{45}$$

which does not agree with (43) for any m, except, of course, for $m = 1$. Moreover, it is independent of the nonlinearity, which first appears in the next order term. However, the range of validity of this expansion also tends to zero in the limit as m tends to zero, where it agrees with the expressions given above for the corresponding energy estimates. Interestingly, this expression coincides with the second-order expansion of the variational estimate of Ponte Castañeda (1991), which is a rigorous upper bound for $\tilde{\sigma}_0$.

5. Results and discussion

Here, results from Section 4 for anti-plane and in-plane loading are presented as a function of the strain-rate-sensitivity m and fiber concentration $c^{(2)}$, for two values of the heterogeneity contrast—one corresponding to stronger fibers ($\sigma_0^{(2)}/\sigma_0^{(1)} = 5$) and the other to weaker fibers ($\sigma_0^{(2)}/\sigma_0^{(1)} = 0.2$). The new "second-order" estimates for the effective flow stress are compared with rigorous bounds and other linearization schemes. For brevity, they will be denoted by the labels SOE(W), SOE(U) and SOE(A), corresponding respectively to the strain-potential formulation (35), the stress-potential formulation (36), and the constitutive-relation (affine) formulation (37). The corresponding "old" second-order estimates of Ponte Castañeda (1996) will be denoted by OSOE(W), OSOE(U) and OSOE(A). Recall that these estimates make use of a similar linear comparison composite except that it uses the tangent moduli of the phases evaluated at the phase averages. The "variational" Hashin–Shtrikman estimates of Ponte Castañeda (1991) provide rigorous upper bounds for all other nonlinear Hashin–Shtrikman estimates, and, in particular, for the second-order estimates. These bounds make use of the secant moduli of the phases evaluated at the second moments of the fields (Suquet, 1995). The Voigt and Reuss estimates are also included for comparison purposes. These are rigorous, microstructure-independent upper and lower bounds, obtained from uniform strain and stress trial fields, respectively.

5.1. Fibers stronger than the matrix

Fig. 2a shows various estimates of the Hashin–Shtrikman type for the effective flow stress of a fiber-reinforced composite, normalized by the flow stress of the matrix, $\tilde{\sigma}_0/\sigma_0^{(1)}$, as a function of the strain-rate-sensitivity m, for a given contrast ($\sigma_0^{(2)}/\sigma_0^{(1)} = 5$) and concentration of fibers ($c^{(2)} = 25\%$). It is observed that all the new estimates (SOE) lie between the variational upper and Reuss lower bounds for all values of m. It is also observed that the old second-order estimates (OSOE) lie higher than the corresponding new estimates (SOE), and, in fact, it can be verified that the OSOE(A) violates the variational upper bound for some values of m close to 1.

Furthermore, while the W- and U-type estimates are different for both the new and the old second-order estimates, it can be seen that the associated duality gap is quite small in general. Moreover, this gap vanishes in the linear case, $m = 1$, where both estimates go to the classical Hashin–Shtrikman estimate, and in the extremely nonlinear rigid-perfectly plastic case, $m = 0$, where both versions go to the Reuss lower bound. Note that as the nonlinearity $n = 1/m$ increases, the reinforcement effect becomes smaller and finally vanishes in the rigid-perfectly plastic limit. Fig. 2b shows the SOE(U) estimates for the normalized effective flow stress as a function of fiber concentration $c^{(2)}$ for several values of the strain-rate-sensitivity ($m = 0, 0.1, 0.2, 1$). (The SOE(W) and SOE(A) estimates are not included in this figure as they are very close to the corresponding SOE(U) estimates.) The main observation here is that the dependence of the effective flow stress on the fiber concentration $c^{(2)}$ becomes progressively weaker with decreasing values of m. However, for very high concentrations, i.e. $c^{(2)} \to 1$, the estimates become very steep as m decreases, and in the limiting case $m = 0$ the estimate presents a jump from 0 to 5.

The dependence on m of the phase averages and fluctuations of the strains associated with the new estimates are shown in Fig. 3a, normalized by the equivalent applied strain $\bar{\varepsilon}_e$. The average strain in the fibers (the stronger phase) can be shown to decay exponentially as $m \to 0$, $\bar{\varepsilon}_e^{(2)} \sim e^{-\alpha/m}$, so that in the ideally plastic limit the average stress in this phase, $\bar{\sigma}_e^{(2)}/\sigma_0^{(2)} \sim \left(\bar{\varepsilon}_e^{(2)}\right)^m \sim O(1)$, remains below the flow stress $\sigma_0^{(2)}$. Recall that the fields were assumed constant inside the fibers, hence there are no fluctuations in phase 2, so that the modulus tensor in the linearized phase is the tangent moduli. The fluctuations in the matrix are seen to

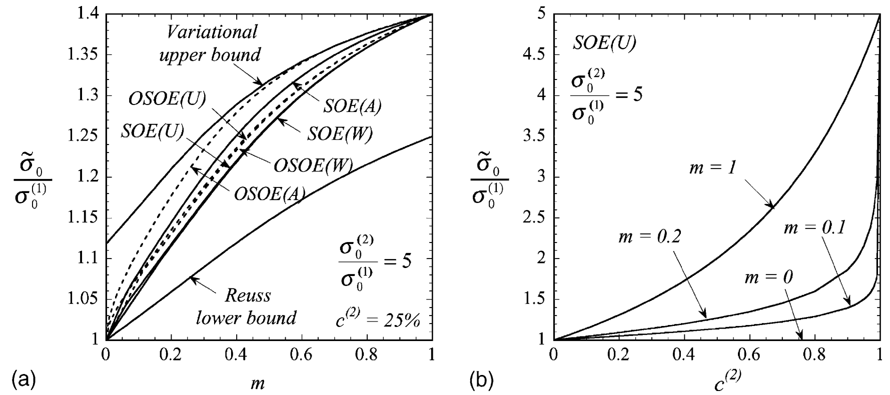

Fig. 2. Effective flow stress normalized by the flow stress of the matrix for a contrast of 5: (a) as a function of the strain-rate-sensitivity for a concentration of 25%; (b) as a function of the fiber concentration for several values of m. Labels 1 and 2 refer to the matrix and fibers, respectively.

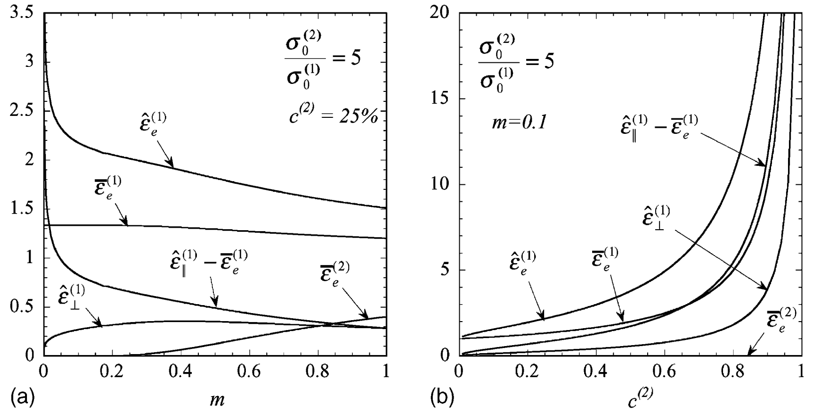

Fig. 3. Phase averages and fluctuations of the strain, normalized by the equivalent applied strain $\bar{\varepsilon}_e$, for a contrast of 5: (a) as a function of the strain-rate-sensitivity for a concentration of 25%; (b) as a function of the fiber concentration for $m = 0.1$. Labels 1 and 2 refer to the matrix and fibers, respectively.

increase with the nonlinearity, meaning the strain field becomes more heterogeneous. Moreover, they actually blow up in the ideally plastic limit ($m = 0$), which is an unexpected result. Note also that these fluctuations are isotropic in the linear case, but become anisotropic as the nonlinearity increases, and they are always higher in the parallel direction. When $m = 0$, the stress–strain curve is "flat", the variables $\bar{\boldsymbol{\varepsilon}}^{(1)}$ and $\hat{\boldsymbol{\varepsilon}}^{(1)}$ become aligned, and since neither of them vanishes, the modulus tensor in the linearized matrix tends to the tangent moduli (which has zero *parallel* component). This is why the new and the old versions of the estimate coincide in this case. The phase averages and fluctuations of the strain for $m = 0.1$ are shown in Fig. 3b as a function of the fiber concentration. For this and smaller values of m, the average strain inside the fibers is almost negligible, except as $c^{(2)} \to 1$, when $\bar{\varepsilon}_e^{(2)}/\bar{\varepsilon}_e \to 1$. Since the fibers practically do not deform, the average strain in the matrix is approximately $\bar{\varepsilon}_e^{(1)}/\bar{\varepsilon}_e \approx 1/c^{(1)}$, which goes to infinity as $c^{(2)} \to 1$. As expected, there are no fluctuations for $c^{(2)} = 0$, since the composite is actually a homogeneous material (the matrix) and hence the fields are constant. As the concentration of fibers increases, the strain field becomes more heterogeneous and thus the fluctuations are higher, and they are seen to blow up when $c^{(2)} \to 1$. But when normalized with the phase average $\bar{\varepsilon}_e^{(1)}$, it can be shown that $\hat{\varepsilon}_{\parallel}^{(1)}/\bar{\varepsilon}_e^{(1)} \to$ const. and $\hat{\varepsilon}_{\perp}^{(1)}/\bar{\varepsilon}_e^{(1)} \to$ const. in this limit.

Fig. 4a shows the corresponding phase averages and fluctuations of the stress normalized by the flow stresses of the phases, as functions of m. The equivalent applied stress has been set equal to the flow stress of the matrix, i.e. $\bar{\sigma}_e = \sigma_0^{(1)}$. Since the stress–strain curve "flattens" as m decreases, and the strain in the matrix does not vanish (see Fig. 3a), the stress fluctuations become smaller, meaning the stress field becomes more homogeneous. Note that, unlike the strain field, the stress field has higher fluctuations in the perpendicular direction. Again, the stress fluctuations are isotropic in the linear case and anisotropic for general values of m, but they vanish when $m = 0$, i.e. the stress field becomes constant. The variables $\bar{\boldsymbol{\sigma}}^{(1)}$ and $\hat{\boldsymbol{\sigma}}^{(1)}$ are the same in this limit, and so the compliance tensor of the linearized matrix becomes the tangent compliance. As already mentioned, the average stress in the fibers (the stronger phase) remains below the flow stress $\sigma_0^{(2)}$ for all values of m. The phase averages and fluctuations of the stress for $m = 0.1$ (continuous lines) can be seen in Fig. 4b, as a function of the fiber concentration. When $c^{(2)} = 0$ the stress fluctuations in the matrix vanish, and are seen to increase with concentration in both directions. Note that for $m = 0$ the average stresses remain below the corresponding flow stresses, except when $c^{(2)} = 0$, where the average stress in the matrix (weaker phase) reaches the flow stress.

At this point, some comments about the rigid-perfectly plastic limit ($m = 0$) are appropriate. For definiteness, the comments will be made in the specific context of anti-plane strain loading, which is easier to

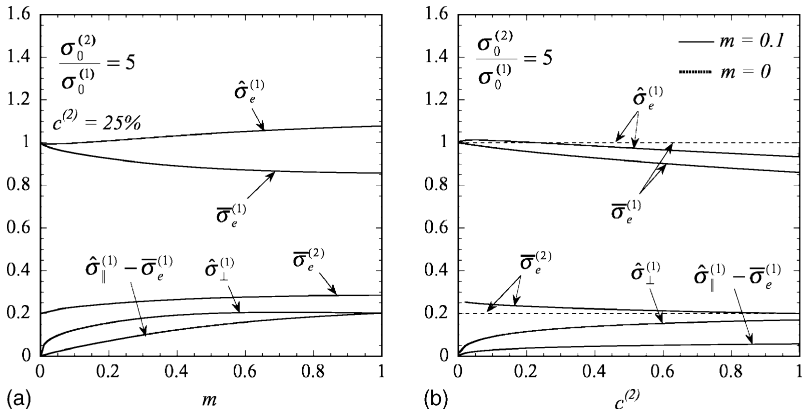

Fig. 4. Phase averages and fluctuations of the stress, normalized by the flow stress $\sigma_0^{(r)}$ of the corresponding phase, for a contrast of 5: (a) as a function of the strain-rate-sensitivity for a concentration of 25%; (b) as a function of the fiber concentration for two values of m. The equivalent applied stress $\bar{\sigma}_e$ has been set equal to $\sigma_0^{(1)}$. Labels 1 and 2 refer to the matrix and fibers, respectively.

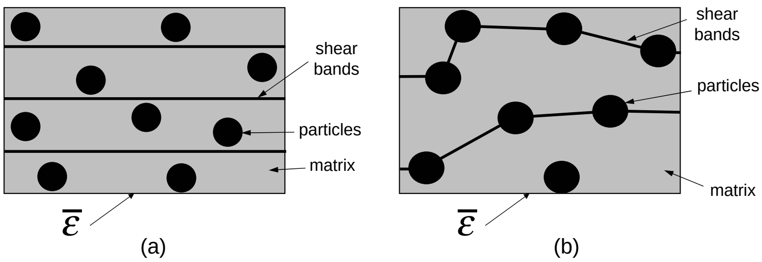

Fig. 5. Rigid-perfectly plastic composite subject to antiplane shear in the vertical direction: (a) when the fibers are stronger than the matrix the shear bands go through the matrix; (b) when the fibers are weaker than the matrix the shear bands go through the fibers.

visualize than the corresponding plane-strain case. First, there is no duality gap in this limit. Both, SOE(W) and SOE(U) estimates give no reinforcement effect due to the presence of stronger fibers (see Fig. 2), except when $c^{(2)} \to 1$. The solution actually reduces to that of rigid particles, regardless of the heterogeneity contrast. It is known from the work of Drucker (1966) that in this case the exact solution corresponds to straight shear bands passing through the matrix, the weaker phase, at least at low concentrations of fibers (see Fig. 5a). The deformation is localized in these bands, which correspond to discontinuities in the displacement field. Note that the results of Fig. 3a, which shows that the average strain in the fibers is zero when $m = 0$, are consistent with such a deformation mechanism. This means that the average stress in the fibers is below their flow stress, i.e. $\bar{\sigma}_e^{(2)} < \sigma_0^{(2)}$, whereas in the matrix, in order to deform, the average stress should be the flow stress, i.e. $\bar{\sigma}_e^{(1)} = \sigma_0^{(1)}$ (see Fig. 4b). Vanishing strain fluctuations in the perpendicular direction are also consistent with the fact that the shear bands are straight, though it is not clear yet what are the implications of infinite strain fluctuations in the parallel direction (see Fig. 3a). It might be related to the presence of not one but an infinite number of bands: one for every "parallel" straight path free of inclusions. Anyway, they do not affect the final expression for the effective flow stress. Vanishing stress fluctuations in the perpendicular direction (see Fig. 4a) means that the field is constant in this direction, namely zero since the fields are aligned with the applied stress, and so the load is entirely carried by the parallel component of the stress, which is also constant and equal to the flow stress.

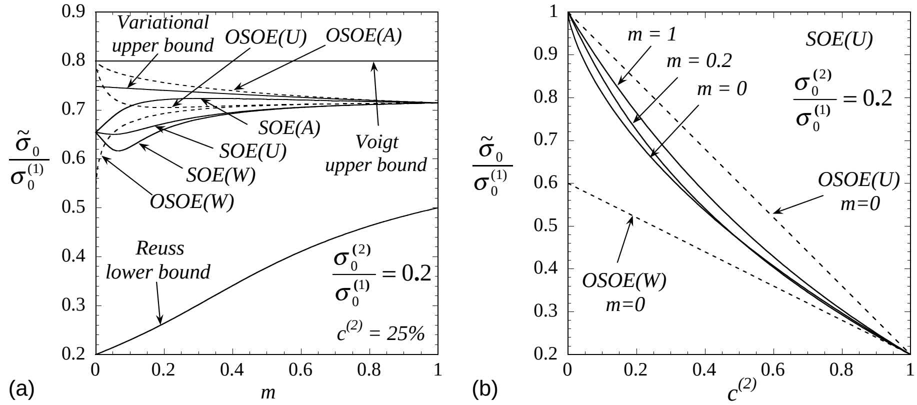

Fig. 6. Effective flow stress normalized by the flow stress of the matrix for a contrast of 0.2: (a) as a function of the strain-rate-sensitivity for a concentration of 25%; (b) as a function of the fiber concentration for three values of *m*. Labels 1 and 2 refer to the matrix and fibers, respectively.

The shear band scheme in Fig. 5a becomes unrealistic for large concentrations, since it should be difficult to find a straight path between fibers. In fact, when $c^{(2)} = 1$, the estimate for the effective property has a discontinuity, and it jumps from 1 to 5 (see Fig. 2b). This may be related to the special microstructure associated with Hashin–Shtrikman estimates. Indeed, Ponte Castañeda (2002b) found a nonvanishing strengthening effect for nonzero values of the concentration, when the self-consistent estimate is used for the linear comparison composite.

5.2. Fibers weaker than the matrix

Estimates of the Hashin–Shtrikman type for a fiber-weakened composite are shown in Fig. 6a as a function of the strain-rate-sensitivity, for a given contrast ($\sigma_0^{(2)}/\sigma_0^{(1)} = 0.2$) and concentration of fibers ($c^{(2)} = 25\%$). The new second-order estimates lie between the bounds for all values of *m* here as well. On the other hand, the old estimate OSOE(U) violates the variational upper bound for sufficiently small values of *m*, and it tends to the Voigt upper bound in the rigid-perfectly plastic limit. Note also that the estimate OSOE(A) violates the variational upper bound for all values of $m < 1$. Moreover, unlike the OSOE(W) and OSOE(U) estimates, which diverge in the rigid-perfectly plastic limit, the SOE(W), SOE(U) and SOE(A) estimates coincide: there is no duality gap in this highly nonlinear limit, for *any* contrast and concentration of fibers. This was already noted in the case of voids by Ponte Castañeda (2002b). However, the SOE estimates still exhibit a nonvanishing duality gap for small, nonzero values of *m*. Of the three possible types of estimates, the stress-potential-type estimates SOE(U) appear to give the best overall predictions in general. (This is because the estimates SOE(W) and SOE(A) exhibit unreasonable dependences on $c^{(2)}$, near $c^{(2)} = 0$, for small, but nonzero values of *m*.) Fig. 6b shows SOE(U) estimates for the normalized effective flow stress as a function of fiber concentration, for several values of the strain-rate-sensitivity ($m = 1,0.2,0$). The new estimate for the rigid-perfectly plastic limit is given by expression (42). Note that the old second-order estimates OSOE(W) and OSOE(U) for $m = 0$ (dashed lines) depend linearly on $c^{(2)}$, and they are considerably different. On the other hand, the new estimates SOE(U) and SOE(W) are equivalent for $m = 0$, and they exhibit a more complex, nonlinear dependence on $c^{(2)}$.

The associated phase averages and fluctuations of the strain, normalized by the applied equivalent strain $\bar{\varepsilon}_e$, are shown in Fig. 7a as a function of the strain-rate-sensitivity. The fields were assumed constant inside the inclusions, so there are no fluctuations in phase 2. As in the previous case, the average strain in the

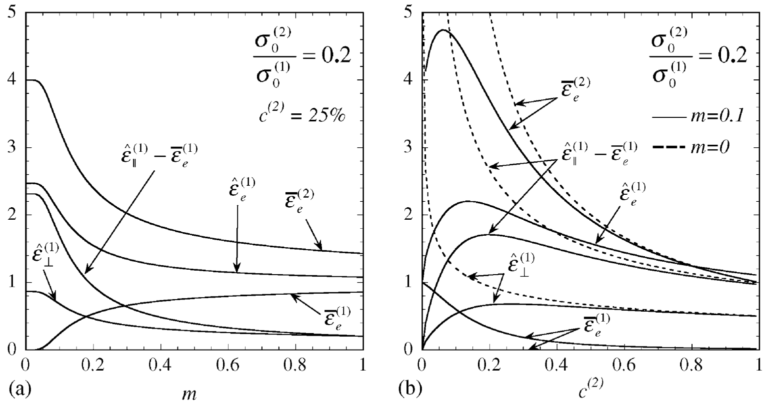

Fig. 7. Phase averages and fluctuations of the strain, normalized by the equivalent applied strain $\bar{\varepsilon}_e$, for a contrast of 5: (a) as a function of the strain-rate-sensitivity for a concentration of 25%; (b) as a function of the fiber concentration for two values of m. Labels 1 and 2 refer to the matrix and fibers, respectively.

stronger phase, now the matrix, goes to zero exponentially as $m \to 0$, $\bar{\varepsilon}_e^{(1)} \approx e^{-\alpha/m}$, such that the average stress in that phase, $\bar{\sigma}_e^{(1)}/\sigma_0^{(1)} \approx \left(\bar{\varepsilon}_e^{(1)}\right)^m \sim O(1)$ in the rigid-perfectly plastic limit. The fluctuations in both directions go up with decreasing m, but they saturate, reaching a maximum value for $m = 0$. They are isotropic for the linear case, becoming more anisotropic with increasing nonlinearity $n = 1/m$, with the parallel strain fluctuations always higher than the perpendicular ones. Fig. 7b shows the normalized phase averages and fluctuations of the strain as a function of concentration, for two values of m (0, 0.1). When $m = 0.1$ (continuous lines), the average strain in the matrix decreases monotonically with increasing concentration of fibers, but in the fibers the average strain has a maximum for some small value of $c^{(2)}$. The fluctuations in the matrix vanish when $c^{(2)} = 0$ as they should, since the composite is actually a homogeneous material (the matrix) in this case. Notice that the fluctuations reach a maximum value and then decrease with increasing fiber concentration. It is interesting to note that they actually increase monotonically when normalized with the phase average $\bar{\varepsilon}_e^{(1)}$. But for $m = 0$ (dashed lines), the fluctuations in the matrix are seen to decrease monotonically with concentration of fibers, and blow up in the dilute limit, i.e. $c^{(2)} \to 0$.

Fig. 8a shows the corresponding phase averages and fluctuations of the stress normalized by the flow stress of the phases, as a function of the strain-rate-sensitivity. The equivalent applied stress has been set equal to the effective flow stress for the rigid-perfectly plastic case, i.e. $\bar{\sigma}_e = \tilde{\sigma}_0(m = 0)$, where $\tilde{\sigma}_0$ is given by (42). As before, the stress fluctuations are isotropic for the linear case, and the anisotropy increases with decreasing m, though this time they do not vanish for $m = 0$. Note that they are higher in the perpendicular direction for all values of m. The average stress in the matrix is always below the flow stress $\sigma_0^{(1)}$, whereas the average stress in the fibers is always above the flow stress $\sigma_0^{(2)}$, except for $m = 0$ where $\bar{\sigma}_e^{(2)} = \sigma_0^{(2)}$. In Fig. 8b we can see the stresses as a function of the concentration, for two values of m (0, 0.1). Again, we observe that the stress fluctuations vanish when $c^{(2)} = 0$, and they increase monotonically (in both directions) with the concentration of fibers.

Some interesting observations can be made for the rigid-perfectly plastic limit ($m = 0$). As in the case of stronger fibers, there is no duality gap in this limit, for any contrast and concentration of fibers. Since now it is the average strain in the matrix that goes to zero, the stress in this phase can take any value from zero to the flow stress, $\sigma_0^{(1)}$, and so we should expect a more complicated stress field than in the case of stronger fibers. Moreover, since $\hat{\boldsymbol{\varepsilon}}^{(1)} \neq \bar{\boldsymbol{\varepsilon}}^{(1)}$, the moduli tensor of the linearized matrix is not the tangent moduli, i.e. $\mathbf{L}_0^{(1)} \neq \mathbf{L}_t^{(1)}$, and that is why the new and the old second-order estimates do not coincide in this case. Although the average strain in the matrix is zero, the matrix does deform—through the strain fluctuations!

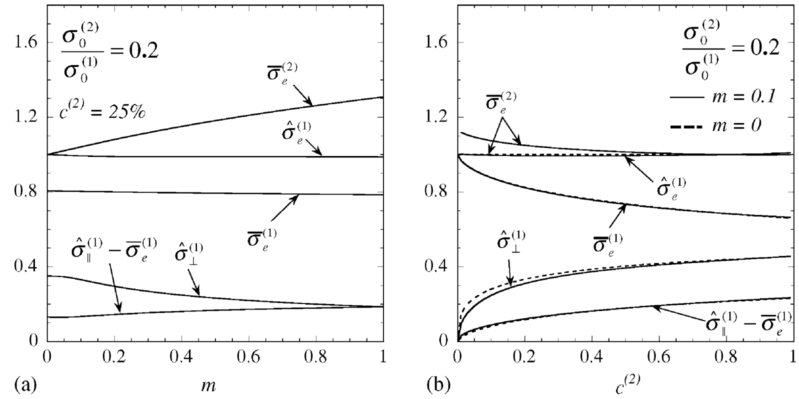

Fig. 8. Phase averages and fluctuations of the stress, normalized by the flow stress $\sigma_0^{(r)}$ of the corresponding phase, for a contrast of 5: (a) as a function of the strain-rate-sensitivity for a concentration of 25%; (b) as a function of the fiber concentration for several values of m. The equivalent applied stress $\bar{\sigma}_e$ has been set equal to the value of $\tilde{\sigma}_0$ at $m = 0$. Labels 1 and 2 refer to the matrix and fibers, respectively.

Fig. 6b shows that there is a weakening effect due to weaker fibers. Note that the effective flow has an infinite slope at zero fiber concentration. In fact, the dilute expansion of expression (42) can be shown to be:

$$\frac{\tilde{\sigma}_0}{\sigma_0^{(1)}} = 1 - \frac{3}{2}\left(1 - \frac{\sigma_0^{(2)}}{\sigma_0^{(1)}}\right)^{4/3}\left(\frac{c^{(2)}}{2}\right)^{2/3}, \tag{46}$$

which has an infinite derivative at $c^{(2)} = 0$. As first suggested by Drucker (1966), when the inclusions are weaker than the matrix the shear bands tend to go through the inclusions (see Fig. 5b). This deformation mechanism for a *periodic three-dimensional* porous medium with a dilute concentration of spherical pores leads to a prediction for the effective flow stress proportional to $1 - \alpha(c^{(2)})^{(2/3)}$. On the other hand, for *periodic two-dimensional* porous media with dilute concentrations of cylindrical pores, Drucker obtained a similar expression, but with an exponent of 1/2, instead of 2/3. The second-order estimates generated in this work predict an exponent of 2/3 for the case of *randomly* distributed cylindrical voids. This is different from Drucker's prediction, but it is not clear at this stage what the effect of randomness versus periodicity of the microstructure is on this exponent. However, recent numerical simulations of porous media based on limit analysis, suggest that the exponent should be between 1/2 and 2/3 (Pastor and Ponte Castañeda, 2002). These simulations consist of finite element discretizations of a hollow cylinder, a commonly used model for porous media, subject to two different types of boundary conditions. Results corresponding to uniform stress lead to the lower exponent, whereas uniform strain results seem to be consistent with a 2/3 exponent. In any event, the important thing to realize is that the exponent would be expected to be less than 1, because of the strong interactions between inclusions, due to the shear bands, even at very low concentrations.

Furthermore, the stress and strain fields exhibit peculiar behaviors in this limit. Fig. 8b shows that as $c^{(2)} \to 0$, the average stress tends to the corresponding value of the flow stress in the given phase, not only in the weaker but also in the stronger phase. Since the stress cannot be higher than the flow stress for $m = 0$, this implies that the stress fields become uniform in both phases. In fact, the stress fluctuations go to zero like $(c^{(2)})^{2/3}$ and $(c^{(2)})^{1/3}$, in the parallel and perpendicular directions, respectively. On the other hand, Fig. 7b shows that the strain fluctuations blow up as $c^{(2)} \to 0$, which is unexpected. The question arises as to whether the fluctuations really do go to infinity when the material is actually more and more homogeneous, or if this is an artifact of the approximation. The answer to this interesting question, which will be pursued in future work, is probably linked to the strong interactions among the fibers, even in the dilute limit.

6. Concluding remarks

The new version of the second-order method of Ponte Castañeda (2002a) was used to estimate the effective behavior of power-law fibrous composites with arbitrary heterogeneity contrast subject to plane- and anti-plane strain loading conditions. Estimates of the Hashin–Shtrikman type for the macroscopic behavior, along with corresponding estimates for the strain and stress fluctuations, were presented and discussed. The new estimates improve on prior estimates arising from an earlier version of the second-order method (that did not incorporate the field fluctuations) in two ways. First, the new estimates, which are exact to second order in the heterogeneity contrast, were found to satisfy rigorous bounds, namely the variational upper bound and the Reuss lower bound. Second, although there is still a difference between the strain and stress-based estimates—the so-called duality gap—it is smaller than for the earlier estimates, and perhaps even more interestingly, is found to vanish in the rigid-perfectly plastic case, for any contrast and concentration of fibers. On the other hand, the field fluctuations, which are known to be isotropic when the material behavior is linear, were found to become progressively more anisotropic as the nonlinearity increases.

Simple expressions for the strongly nonlinear rigid-perfectly plastic limit were derived and studied in detail. The resulting predictions seem to be consistent with deformation mechanisms involving shear bands. In the fiber-reinforced case, this translated into no reinforcement effect, and infinite strain fluctuations were predicted in the matrix. In the case of weaker fibers, the dilute limit shows a dependence of the effective property on the concentration of fibers of the type $\tilde{\sigma}_0/\sigma_0^{(1)} \approx 1 - \alpha(c^{(2)})^{2/3}$, which is not in exact agreement with Drucker's results for periodic media, but it is closer and more realistic than previous estimates. This is a sensitive limit where both phases are at yield, and the strain fluctuations in the matrix blow up. The question remains open as to what are the implications of this result.

The effect of tension along the fibers will be considered in future work in an attempt to generate the yield surface for general loading conditions. The use of self-consistent estimates for the linear comparison composite would allow the incorporation of information about the fluctuations in both phases, and the corresponding nonlinear estimates would be expected to be more accurate for high concentration of inclusions, at least for certain types of symmetric microstructures. This problem will also be addressed in future work.

Acknowledgements

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